7.1.10

(a)

$$\int_c f(x,y,z)\ ds = \int_0^{2\pi} (\sin t + \cos t + t)\sqrt{\sin^2 t + \cos^2 t + 1^2}\ dt =$$

$$= \int_0^{2\pi} \sqrt{2}(\sin t + \cos t + t)\ dt = 2\sqrt{2}\pi^2$$

(b)

$$\int_c f(x,y,z)\ ds = \int_0^{2\pi} \cos t\sqrt{\sin^2 t + \cos^2 t + 1^2}\ dt =$$

$$= \int_0^{2\pi} \sqrt{2}\cos t\ dt = 0$$

7.1.14

(a) We have $c(\theta) = (r\cos\theta, r\sin\theta)$ where r is a function of θ, hence $c'(\theta) = (r'\cos\theta - r\sin\theta, r'\sin\theta + r\cos\theta)$ and finally

$$c'(\theta) = \sqrt{(r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2} = \sqrt{(r')^2 + r^2}$$

Substituting this into the path integral gives the desired result.

(b) The arclength is just the path integral of the function 1. This gives

$$l(c) = \int_0^{2\pi} \sqrt{r^2 + (r')^2}\ d\theta = \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\ d\theta =$$

$$= \int_0^{2\pi} \sqrt{2 + 2\cos\theta}\ d\theta$$

Plugging $\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2)$ we get $2 + 2\cos\theta = 4\cos^2(\theta/2)$, which plugging it back into the integral gives that the arclength is 8.

7.1.17

Using the expression given above we have

$$\bar{y} = \frac{\int_c y(x,y,z)\ ds}{l(c)} = \frac{\int_0^\pi a\sin\theta\sqrt{(-a\cos\theta)^2 + (a\sin\theta)^2}\ ds}{a\pi}$$

Which gives $\bar{y} = \dfrac{2a^2}{a\pi} = \dfrac{2}{\pi}a.$

7.2.3

(a)

$$\int_0^1 (t,t,t)\cdot(1,1,1)\ dt = \int_0^1 3t\ dt = 3/2$$

(b)

$$\int_0^{2\pi} (\cos t, \sin t, 0)\cdot(-\sin t, \cos t, 0)\ dt = \int_0^{2\pi} 0\ dt = 0$$

(c)

$$\int_0^{2\pi} (\sin t, 0, \cos t)\cdot(\cos t, 0, -\sin t)\ dt = \int_0^{2\pi} 0\ dt = 0$$

(d)

$$\int_{-1}^{2} (t^2, 3t, 2t^3)\cdot(2t, 3, 6t^2)\ dt = \int_{-1}^{2} 2t^3 + 9t + 12t^5\ dt = 147$$

7.2.4

Using that if $x(t)$ then $dx = x'(t)\ dt$ we have:

(a)

$$\int_c x\ dy - y\ dx = \int_0^{2\pi} \cos t(\cos t\ dt) - \sin t(-\sin t\ dt) = \int_0^{2\pi} 1\ dt = 2\pi$$

(b)

$$\int_c x\ dy + y\ dx = \int_0^{2\pi} \cos t(\cos t\ dt) + \sin t(-\sin t\ dt) = \int_0^{2\pi} \cos(2t)\ dt = 0$$

(c) We split the integral in two, each of them consisting of each segment. The first one is parametrized as $(1-t, t, 0)$ and the other one as $(0, 1-t, t)$, both with $t \in [0,1]$.

First segment:

$$\int_c yz\ dx + xz\ dy + xy\ dz = \int_0^1 t(0)(-1)\ dt + (1-t)(0)(1)\ dt + (1-t)t(0)\ dt = 0$$

Second segment:

$$\int_c yz\ dx + xz\ dy + xy\ dz = \int_0^1 (1-t)(t)(0)\ dt + (0)(t)(-1)\ dt + (0)(1-t)(1)\ dt = 0$$

And hence the total integral is 0.

(d) Now our path is $(t, 0, t^2)$ with $t \in [-1, 1]$:

$$\int_c x^2\ dx - xy\ dy + dz = \int_{-1}^1 t^2\ dt + 0\ dt + 2t\ dt = 2/3$$

7.3.18

We have that

$$f(1,1,2) - f(0,0,0) = \int_c f \cdot ds$$

where c is any path from the origin to $(1,1,2)$. For instance we can take $c(t) = (t, t, 2t)$. In this case

$$\int_c f \cdot ds = \int_0^1 (4t^3 + 4t)e^{t^2}\ dt$$

Which has as primitive $2t^2 e^{t^2}$. This gives that the path integral is $2e$ and hence $f(1,1,2) = 5 + 2e$.

7.4.14

The partial derivatives of the parametrization at $(1,1)$ give the vectors $T_u = (2,0,2)$ and $T_v = (0,2,2)$. This means that the normal vector is parallel to $T_u \times T_v = (-4,-4,4)$ and thus we can pick $(-1,-1,1)$. Since the plane has to pass through the point $(1,1,2)$ we have that the plane is given by

$$(x-1, y-1, z-2) \cdot (-1,-1,1) = 0 \iff x + y - z = 0$$

7.4.20

(a) The range of the a matrix is the vector space spanned by its columns, which are precisely T_u and T_v.

(b) Since we are in a 3-dimensional vector space, we have that w is perpendicular to a given vector if and only if it lies in its perpendicular plane. In our case, the plane perpendicular to $T_u \times T_v$ is spanned by T_u and T_v, which by (a) means that w is perpendicular if and only if it is in the range of $D\Phi(u_0, v_0)$.

(c) The tangent plane can be parametrized as a point inside plus any combination of two spanning vectors. This means that a valid parametrization would be

$$\Phi(u_0, v_0) + (u - u_0)T_u + (v - v_0)Tv$$

Which written in matricial form gives the desired result.

7.4.5

(a)

$$T_u = (e^u \cos v, e^u \sin v, 0)$$
$$T_v = (-e^u \sin v, e^u \cos v, 1)$$

And hence

$$T_u \times T_v = (e^u, e^u, e^{2u})$$

(b) In this case $T_u = (0,1,0)$ and $T_v = (-1,0,1)$. We have $T_u \times T_v = (1,0,1)$ and that the plane passes through $(0,1,\pi/2)$. Hence the equation of the plane is

$$(x, y, z - \pi/2) \cdot (1, 0, 1) = 0 \Longleftrightarrow x + z = \pi/2$$

(c)

$$A(D) = \int\int_D \|T_u \times T_v\| \; dv \; du = \int_0^1 \int_0^\pi e^u \sqrt{2 + e^{2u}} \; dv \; du =$$

$$= \pi \int_0^1 e^u \sqrt{2 + e^{2u}} \; dv$$

Which is rather hard to compute.

7.4.10

We will use the standard spherical parametrization with constant radius one. Note that the fact that being inside the cone translates into $\phi \in [0, \pi/4]$.

Then we have that this portion of the sphere is given by

$$\Phi(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

This gives

$$T_\phi = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi)$$
$$T_\theta = (-\sin\phi\sin\theta, \sin\phi, \cos\theta, 0)$$

And thus

$$T_\phi \times T_\theta = (\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \cos\phi\sin\phi)$$

With norm

$$\|T_\phi \times T_\theta\| = |\sin\phi|$$

Hence

$$A(\Phi) = \int_0^{\pi/4} \int_0^{2\pi} |\sin\phi| \; d\theta \; d\phi = 2\pi\left(1 - \frac{\sqrt{2}}{2}\right)$$